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Units of the Group Algebra of the Group $C_n \times D_8$ Over Any Finite Field of Characteristic 2

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Abstract

This article is concerned with establishing the structure of the unit group of certain group algebras. In particular, we establish the structure of the unit group of the group algebra $\mathbb{F}_{2^k}(C_n \times D_8)$ for $n \geq 1$.

Keywords: Group ring; group algebra; dihedral group; cyclic group.

1 Introduction

Let KG denote the group algebra of the group G over the field K. The set of all the invertible elements of a ring S form a group called the unit group of S, denoted by $\mathcal{U}(S)$. The homomorphism $\varepsilon : KG \longrightarrow K$ given by

$$\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g,$$

is called the augmentation mapping of FG.

The normalized unit group of FG denoted by $V(KG)$ consists of all the invertible elements of KG of augmentation 1. It is a well known fact that $\mathcal{U}(KG) \cong \mathcal{U}(K) \times V(KG)$.

It is well known that $|V(KG)| = |K|^{|G|-1}$ where G is a finite p-group and K is a field of characteristic p. Sandling in [11], provides a basis for $V(\mathbb{F}_pG)$ where G is an abelian p-group and $\mathbb{F}_p$ is the Galois field of p-elements. Let D_8 be the dihedral group of order 8. The structures of $\mathcal{U}(\mathbb{F}_2D_8)$, $\mathcal{U}(\mathbb{F}_{2^k}D_8)$ and $\mathcal{U}(\mathbb{F}_{2^k}(C_2 \times D_8))$ are established in [12, 7, 8] respectively. The structure of $\mathcal{U}(\mathbb{F}_{2^k}D_{2n})$ is established in [9] when n is odd. For an overview of modular group algebras, consult [2].

The map $* : KG \longrightarrow KG$ defined by,

$$\left(\sum_{g\in G} a_g g\right)^* = \sum_{g\in G} a_g g^{-1},$$

is an antiautomorphism of of KG of order 2. An element v of $V(KG)$ satisfying $v^{-1} = v^*$ is called unitary. We denote by $V_*(KG)$ the subgroup of $V(KG)$ formed by the unitary elements of KG. In [4] a basis for $V_*(KG)$ is constructed for any field of characteristic $p > 2$ and any finite abelian p-group. The structure of $V_*(\mathbb{F}_2G)$ is established in [3] for all groups of order 8 and 16 and the structure of $V_*(\mathbb{F}_2Q_8)$ is established in [6] where Q_8 is the quaternion group of order 8. Additionally, the order of $V_*(\mathbb{F}_{2^k}G)$ is determined for special cases of G in [5]. The structures of $V_*(\mathbb{F}_{2^k}G)$ when $G = M_{16}$ and $G = D_{2n}$ are established in [1] and [10] respectively where $M_{16} = \langle x, y \mid x^8 = y^2 = 1,\, xy = yx^5\rangle$ and D_{2n} is the dihedral group of order $2n$. Our main result is:

Theorem 1.1.

$$\mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8)) \cong \begin{cases} [(C_2^{nk} \times C_4^{nk}) \rtimes (C_2^{nk} \times C_4^{nk})] \rtimes \mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n)) & \textit{when } n \textit{ is odd},\\ [(C_2^{2nk} \times C_4^{\frac{nk}{2}}) \rtimes (C_2^{2nk} \times C_4^{\frac{nk}{2}})] \rtimes \mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n)) & \textit{when } n \textit{ is even}.\end{cases}$$

2 The Structure of $\mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8))$

Let $G = C_n \times D_8 = \langle x, y, z \mid x^4 = y^2 = z^n = 1,\, x^y = x^{-1},\, xz = zx,\, yz = zy\rangle$ where $n \geq 1$. The natural group homomorphism $G \longrightarrow G/\langle x\rangle$ extends linearly to the algebra homomorphism $\theta : \mathbb{F}_{2^k}(C_n \times D_8) \longrightarrow \mathbb{F}_{2^k}(C_2 \times C_n)$, where

$$\sum_{i=1}^{4} x^{i-1}(\alpha_i + \alpha_{i+4}z + \cdots + \alpha_{i+4n}z^{n-1} + \alpha_{i+4n+4}y + \alpha_{i+4n+8}yz + \cdots + \alpha_{i+8n}yz^{n-1}) \mapsto$$
$$\sum_{i=1}^{4}(\alpha_i + \alpha_{i+4}b + \cdots + \alpha_{i+4n}b^{n-1} + \alpha_{i+4n+4}a + \alpha_{i+4n+8}ab + \cdots + \alpha_{i+8n}ab^{n-1}),$$

and $C_2 \times C_n = \langle a, b \,|\, a^2 = b^{n-1} = 1,\, ab = ba\rangle$. If we restrict θ to $\mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8))$, we can construct the group epimorphism $\theta' : \mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8)) \longrightarrow \mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n))$. Consider the group homomorphism $\psi : \mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n)) \longrightarrow \mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8))$ by,

$$\gamma_1 + \gamma_2 b + \cdots + \gamma_n b^{n-1} + \delta_1 a + \delta_2 ab + \cdots + \delta_n ab^{n-1} \mapsto$$
$$\gamma_1 + \gamma_2 z + \cdots + \gamma_n z^{n-1} + \delta_1 y + \delta_2 yz + \cdots + \delta_n yz^{n-1},$$

where $\gamma_i, \delta_j \in \mathbb{F}_{2^k}$. Clearly, $\theta' \circ \psi$ is the identity map of $\mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n))$. Therefore, $\mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8))$ is a split extension of $\mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n))$ by $ker(\theta')$ and $\mathcal{U}(\mathbb{F}_{2^k}(C_n \times D_8)) \cong H \rtimes \mathcal{U}(\mathbb{F}_{2^k}(C_2 \times C_n))$ where $H \cong ker(\theta')$.

Lemma 2.1. *H has exponent 4.*

Proof. Let $h = 1 + \sum_{j=1}^{n} A_j + \sum_{k=1}^{n} B_k y \in H$ where

$$A_j = \sum_{i=1}^{3} \gamma_{i+3(j-1)} z^{j-1}(1 + x^i) \text{ and } B_k = \sum_{i=1}^{3} \gamma_{i+3(k+n-1)} z^{k-1}(1 + x^i),$$

and $\gamma_j \in \mathbb{F}_{2^k}$. Then,

$$\begin{aligned} h^2 &= 1 + \sum_{j=1}^{n}\sum_{k=1}^{n} A_j A_k + \sum_{j=1}^{n}\sum_{k=1}^{n} A_j B_k y + \sum_{j=1}^{n}\sum_{k=1}^{n} B_j y A_k + \sum_{j=1}^{n}\sum_{k=1}^{n} B_j y B_k y \\ &= 1 + \sum_{j=1}^{n}\sum_{k=1}^{n} (A_j A_k + B_j B_k') + \sum_{j=1}^{n}\sum_{k=1}^{n} (A_j B_k + B_j A_k')\, y, \end{aligned}$$

where $yA_k = A_k' y$ and $yB_k = B_k' y$. Now,

$$\begin{aligned} A_j A_k &= \sum_{i=1}^{3} \gamma_{i+3(j-1)} z^{j-1}(1 + x^i) \sum_{i=1}^{3} \gamma_{i+3(k-1)} z^{k-1}(1 + x^i) \\ &= z^{j+k-2}(\gamma_{3j-2}\gamma_{3k-2} + \gamma_{3j}\gamma_{3k})(1 + x^2) + (\gamma_{3j-2}\gamma_{3k-1} + \gamma_{3j-1}\gamma_{3k-2} \\ &\quad + \gamma_{3j-1}\gamma_{3k} + \gamma_{3j}\gamma_{3k-1})\hat{x} + (\gamma_{3j-2}\gamma_{3k} + \gamma_{3j}\gamma_{3k-2})(x + x^3), \end{aligned}$$

$$\begin{aligned} B_j B_k' &= B_j y B_k y \\ &= \sum_{i=1}^{3} \gamma_{i+3(j+n-1)} z^{j-1}(1 + x^i) \sum_{i=1}^{3} \gamma_{i+3(k+n-1)} z^{k-1}(1 + x^{4-i}) \\ &= z^{j+k-2}(\gamma_{3j+3n-2}\gamma_{3k+3n-2} + \gamma_{3j+3n}\gamma_{3k+3n})(x + x^3) + (\gamma_{3j+3n-2}\gamma_{3k+3n-1} \\ &\quad + \gamma_{3j+3n-1}\gamma_{3k+3n-2} + \gamma_{3j+3n-1}\gamma_{3k+3n} + \gamma_{3j+3n}\gamma_{3k+3n-1})\hat{x} \\ &\quad + (\gamma_{3j+3n-2}\gamma_{3k+3n} + \gamma_{3j+3n}\gamma_{3k+3n-2})(1 + x^2), \end{aligned}$$

and

$$\begin{aligned} A_jB_k + B_jA'_k &= \sum_{i=1}^{3}\gamma_{i+3(j-1)}z^{j-1}(1+x^i)\sum_{i=1}^{3}\gamma_{i+3(k+n-1)}z^{k-1}(1+x^i) \\ &+ \sum_{i=1}^{3}\gamma_{i+3(j+n-1)}z^{j-1}(1+x^i)\sum_{i=1}^{3}\gamma_{i+3(k-1)}z^{k-1}(1+x^{4-i}) \\ &= (\gamma_{3j-2}\gamma_{3k+3n-2} + \gamma_{3j}\gamma_{3k+3n} + \gamma_{3j+3n-2}\gamma_{3k} + \gamma_{3j+3n}\gamma_{3k-2})(1+x^2) \\ &+ (\gamma_{3j-2}\gamma_{3k+3n} + \gamma_{3j}\gamma_{3k+3n-2} + \gamma_{3j+3n-2}\gamma_{3k-2} + \gamma_{3j+3n}\gamma_{3k})(x+x^3) \\ &+ (\gamma_{3j-2}\gamma_{3k+3n-1} + \gamma_{3j-1}\gamma_{3k+3n-2} + \gamma_{3j-1}\gamma_{3k+3n} + \gamma_{3j}\gamma_{3k+3n-1} \\ &+ \gamma_{3j+3n-2}\gamma_{3k-1} + \gamma_{3j+3n-1}\gamma_{3k-2} + \gamma_{3j+3n-1}\gamma_{3k} + \gamma_{3j+3n}\gamma_{3k-1})\hat{x}. \end{aligned}$$

Therefore, h^2 takes the form

$$\begin{cases} 1+(1+x^2)\sum_{i=1}^{\frac{n}{2}}\gamma_i z^{2(n-1)} + (x+x^3)\sum_{i=1}^{\frac{n}{2}}\delta_i z^{2(n-1)} + \hat{x}y\sum_{i=1}^{n}\beta_i z^{n-1} & \text{if } n \text{ is even,} \\ 1+(1+x^2)\sum_{i=1}^{n}\gamma_i z^{n-1} + (x+x^3)\sum_{i=1}^{n}\delta_i z^{n-1} + \hat{x}y\sum_{i=1}^{n}\beta_i z^{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

where $\gamma_i, \delta_j, \beta_k \in \mathbb{F}_{2^k}$. First, let n be even. Clearly, $\hat{x}y = y\hat{x}$, $(1+x^2)\hat{x}y = \hat{x}y(1+x^2)$ and $(x+x^3)\hat{x}y = \hat{x}y(x+x^3)$, therefore,

$$\begin{aligned} h^4 &= 1 + \left[(1+x^2)\sum_{i=1}^{\frac{n}{2}}\gamma_i z^{2(n-1)}\right]^2 + \left[(x+x^3)\sum_{i=1}^{\frac{n}{2}}\delta_i z^{2(n-1)}\right]^2 + \left[\hat{x}y\sum_{i=1}^{n}\beta_i z^{n-1}\right]^2 \\ &= 1 + \left[(1+x^2)\right]^2\left[\sum_{i=1}^{\frac{n}{2}}\gamma_i z^{2(n-1)}\right]^2 + \left[(x+x^3)\right]^2\left[\sum_{i=1}^{\frac{n}{2}}\delta_i z^{2(n-1)}\right]^2 + [\hat{x}y]^2\left[\sum_{i=1}^{n}\beta_i z^{n-1}\right]^2. \end{aligned}$$

Additionally, $(1+x^2)^2 = 1+x^4 = 0$, $(x+x^3)^2 = x^2+x^6 = 0$ and $(\hat{x}y)^2 = (\hat{x})^2y^2 = 0$. Thus, $h^4 = 1$. Similarly $h^4 = 1$ when n is odd.

□

Lemma 2.2. *Let S be the subgroup of H where the elements of H that take the form:*

$$1 + \sum_{j=1}^{n} A_j(1+y),$$

where $A_j = \sum_{i=1}^{3}\gamma_{i+3(j-1)}z^{j-1}(1+x^i)$ and $\gamma_i \in \mathbb{F}_{2^k}$. Then,

$$S \cong \begin{cases} C_2^{nk} \times C_4^{nk} & \text{if } n \text{ is odd,} \\ C_2^{2nk} \times C_4^{\frac{nk}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $s_1 = 1 + \sum_{j=1}^{n}A_j(1+y) \in S$ and $s_2 = 1 + \sum_{k=1}^{n}B_k(1+y) \in S$,

where $A_j = \sum_{i=1}^{3}\gamma_{i+3(j-1)}z^{j-1}(1+x^i)$, $B_k = \sum_{i=1}^{3}\delta_{i+3(k-1)}z^{k-1}(1+x^i)$ and $\gamma_i, \delta_j \in \mathbb{F}_{2^k}$.

Then,

$$s_1 s_2 = 1 + \sum_{j=1}^{n}(A_j + B_j)(1+y) + \sum_{j=1}^{n}\sum_{k=1}^{n} A_j(1+y)B_k(1+y).$$

Now,

$$\begin{aligned} A_j(1+y)B_k(1+y) &= (A_j + A_j y)(B_k + B_k y) \\ &= A_jB_k + A_jB_ky + A_jyB_k + A_jyB_ky \\ &= A_jB_k + A_jB_ky + A_jB_k'y + A_jB_k' \\ &= (A_jB_k + A_jB_k')(1+y) \\ &= A_j(B_k + B_k')(1+y), \end{aligned}$$

where $yB_k = B_k'y$ and

$$\begin{aligned} B_k + B_k' &= \sum_{i=1}^{3}\delta_{i+3(k-1)}z^{k-1}(1+x^i) + \sum_{i=1}^{3}\delta_{i+3(k-1)}z^{k-1}(1+x^{4-i}) \\ &= \sum_{i=1}^{3}\delta_{i+3(k-1)}z^{k-1}x^i + \sum_{i=1}^{3}\delta_{i+3(k-1)}z^{k-1}x^{4-i} \\ &= (\delta_{3k-2} + \delta_{3k})(x + x^3)z^{k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} A_j(1+y)B_k(1+y) &= \sum_{i=1}^{3}\gamma_{i+3(j-1)}z^{j-1}(1+x^i)(\delta_{3k-2}+\delta_{3k})(x+x^3)z^{k-1}(1+y) \\ &= (\delta_{3k-2}+\delta_{3k})\sum_{i=1}^{3}\gamma_{i+3(j-1)}(1+x^i)(x+x^3)z^{j+k-2}(1+y) \\ &= (\delta_{3k-2}+\delta_{3k})(a_{3j-2}+a_{3j})\hat{x}z^{j+k-2}(1+y) \\ &= (\gamma_{3j-2}+\gamma_{3j})(\delta_{3k-2}+\delta_{3k})z^{j+k-2}\hat{x}(1+y). \end{aligned}$$

It follows that S is closed. Clearly $A_j(1+y)B_k(1+y) = B_k(1+y)A_j(1+y)$ for all $j,k \in \{1,2,\dots,n\}$, therefore S is abelian. It remains to construct the structure of S. Let $s = 1 + \sum_{i=1}^{3}(\gamma_i + \gamma_{i+3}z + \gamma_{i+6}z^2 + \cdots + \gamma_{i+3(n-1)}z^{n-1})(1+x^i)(1+y) \in S$ where $\gamma_i \in \mathbb{F}_{2^k}$. When n is odd,

$$s^2 = 1 + \sum_{i=1}^{n}(\gamma_{3i-2}+\gamma_{3i})^2 z^{2(i-1)}\hat{x}(1+y).$$

Therefore, $s^2 = 1$ iff $\gamma_{3i-2} = \gamma_{3i}$ for $1 \le i \le n$. Clearly the number of elements of S of order 2 or 1 is equal to 2^{2nk} and the number of elements of S of order 4 is equal to $2^{3nk} - 2^{2nk} = 2^{2nk}(2^{nk}-1)$. On the other hand, $S \cong C_2^l \times C_4^m$ for some l and m. The number of elements of $C_2^l \times C_4^m$ of order 2 or 1 is equal to $2^l \cdot 2^m = 2^{l+m}$. Additionally, the number of elements of order 4 in $C_2^l \times C_4^m$ is equal to $2^l \cdot 4^m - 2^{l+m} = 2^{l+m}(2^m - 1)$. Clearly $m = l = nk$.

When n is even,

$$s^2 = 1 + \sum_{i=1}^{\frac{n}{2}} \delta_i z^{2(i-1)} \hat{x}(1+y),$$

where $\delta_j = \left(\gamma_{(3j-2)} + \gamma_{(3j)} + \gamma_{\left(3j-2+\frac{3n}{2}\right)} + \gamma_{\left(3j+\frac{3n}{2}\right)}\right)^2$. The number of elements of S of order 2 or 1 is equal to $\left(2^{5k}\right)^{\frac{n}{2}} = 2^{\frac{5nk}{2}}$ and the number of elements of S of order 4 is equal to $2^{3nk} - 2^{\frac{5nk}{2}} = 2^{\frac{5nk}{2}}\left(2^{\frac{nk}{2}} - 1\right)$. Therefore, $m = \dfrac{nk}{2}$ and $l = 2nk$.

□

Lemma 2.3. *Let N be the normal subgroup of H where the elements of H that take the form:*

$$1 + \sum_{i=1}^{n} z^{i-1}\left(\alpha_i(x+x^3) + \left[(\beta_i+\gamma_i)(1+x^3) + \gamma_i\hat{x}\right] y\right),$$

where $\gamma_i, \beta_j, \gamma_k \in \mathbb{F}_{2^k}$. Then,

$$N \cong \begin{cases} C_2^{nk} \times C_4^{nk} & \text{if } n \text{ is odd,} \\ C_2^{2nk} \times C_4^{\frac{nk}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $n_1 = 1 + \sum_{i=1}^{n} C_i \in N$ and $n_2 = 1 + \sum_{i=1}^{n} D_i \in N$, where

$$C_i = z^{i-1}\left(\alpha_i(x+x^3) + \left[(\beta_i+\gamma_i)(1+x^3) + \gamma_i\hat{x}\right] y\right),$$

$$D_j = z^{j-1}\left(\delta_j(x+x^3) + \left[(\mu_j+\nu_j)(1+x^3) + \nu_j\hat{x}\right] y\right),$$

and $\gamma_i, \beta_j, \gamma_k, \delta_r, \mu_s, \nu_t \in \mathbb{F}_{2^k}$. Then,

$$n_1 n_2 = 1 + \sum_{i=1}^{n}(C_i + D_i) + \sum_{j=1}^{n}\sum_{k=1}^{n} C_j D_k.$$

Now,

$$\begin{aligned}
C_j D_k &= z^{j+k-2}\left[\ \left(\alpha_j(x+x^3) + \left[(\beta_j+\gamma_j)(1+x^3) + \gamma_j\hat{x}\right] y\right) \times \right. \\
&\left. \left(\delta_k(x+x^3) + \left[(\mu_k+\nu_k)(1+x^3) + \nu_k\hat{x}\right] y\right)\ \right] \\
&= z^{j+k-2}\left[\ \alpha_j\delta_k(x+x^3)^2 + \alpha_j(x+x^3)\left[(\mu_k+\nu_k)(1+x^3) + \nu_k\hat{x}\right] y \right. \\
&+ \delta_k(1+x^2)\left[(\beta_j+\gamma_j)(1+x^3) + \gamma_j\hat{x}\right] y + (\beta_j+\gamma_j)(\mu_k+\nu_k)(1+x^3)(1+x) \\
&\left. + (\beta_j+\gamma_j)\nu_k(1+x^3)\hat{x} + \gamma_j(\mu_k+\nu_k)(1+x)\hat{x} + \gamma_j\nu_k(\hat{x})^2\ \right] \\
&= z^{j+k-2}\left[\ \alpha_j(\mu_k+\nu_k)(x+x^3)(1+x^3)y + \delta_k(\beta_j+\gamma_j)(1+x^2)(1+x^3)y \right. \\
&\left. + (\beta_j+\gamma_j)(\mu_k+\nu_k)(x+x^3)\ \right] \\
&= z^{j+k-2}\left[\ (\beta_j+\gamma_j)(\mu_k+\nu_k)(x+x^3) + (\alpha_j(\mu_k+\nu_k) + \delta_k(\beta_j+\gamma_j))\hat{x}y\ \right].
\end{aligned}$$

Clearly, N is closed and $C_j D_k = D_k C_j$ for all $j, k \in \{1, 2, \ldots, n\}$.

Let $n = 1 + \sum_{i=1}^{n} C_i \in N$ where

$$C_i = z^{i-1}\left(\alpha_i(x+x^3) + \left[(\beta_i+\gamma_i)(1+x^3) + \gamma_i\hat{x}\right]y\right),$$

and $\gamma_i, \beta_j, \gamma_k \in \mathbb{F}_{2^k}$. Then,

$$n^2 = \begin{cases} 1 + \sum_{i=1}^{n}(\beta_i+\gamma_i)^2(x+x^3)z^{i-1} & \text{if } n \text{ is odd,} \\ 1 + \sum_{i=1}^{\frac{n}{2}}(\beta_i+\gamma_i+\beta_{i+\frac{n}{2}}+\gamma_{i+\frac{n}{2}})^2(x+x^3)z^{2(i-1)} & \text{if } n \text{ is even.} \end{cases}$$

Clearly, for all $i \in \{1, 2, \dots, n\}$, $n^2 = 1$ if and only if $\beta_i = \gamma_i$ when n is odd and $\beta_i + \gamma_i + \beta_{i+\frac{n}{2}} + \gamma_{i+\frac{n}{2}} = 0$ when n is even.

Therefore,

$$N \cong \begin{cases} C_2^{nk} \times C_4^{nk} & \text{if } n \text{ is odd,} \\ C_2^{2nk} \times C_4^{\frac{nk}{2}} & \text{if } n \text{ is even.} \end{cases}$$

It remains to prove that N is normal. Let n be odd, $s = 1 + \sum_{j=1}^{n} A_j(1+y)$ where $A_j = \sum_{i=1}^{3}\delta_{i+3(j-1)}z^{j-1}(1+x^i)$ and $\gamma_i \in \mathbb{F}_{2^k}$ and $n = 1 + \sum_{k=1}^{n} C_k \in N$ where

$$C_k = z^{k-1}\left(\alpha_k(x+x^3) + \left[(\beta_k+\gamma_k)(1+x^3) + \gamma_k\hat{x}\right]y\right),$$

and $\gamma_i, \delta_j, \beta_k, \delta_l \in \mathbb{F}_{2^k}$. Then,

$$\begin{aligned} n^s &= s^{-1}ns \\ &= s^3 ns \\ &= \left(1 + \sum_{j=1}^{n}(A_j + F_j\hat{x})(1+y)\right)\left(1 + \sum_{k=1}^{n} C_k\right)\left(1 + \sum_{l=1}^{n} A_l(1+y)\right), \end{aligned}$$

where $F_j = \lambda_j z^{j-1}$, $\lambda_1 = (\delta_1+\delta_3)^2$, $\lambda_j = (\delta_{3n-2-3(j-2)} + \delta_{3n-3(j-2)})^2$ where $j \in \{2, 3, \dots, n\}$.

Now,

$$\begin{aligned} n^s = 1 &+ \sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y) + \sum_{k=1}^{n} C_k + \sum_{l=1}^{n} A_l(1+y) + \sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{k=1}^{n} C_k \\ &+ \sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{l=1}^{n} A_l(1+y) + \sum_{k=1}^{n} C_k\sum_{l=1}^{n} A_l(1+y) \\ &+ \sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{k=1}^{n} C_k\sum_{l=1}^{n} A_l(1+y). \end{aligned}$$

Now,

$$\sum_{j=1}^{n}(A_j + F_j\hat{x})(1+y)\sum_{k=1}^{n} C_k\sum_{l=1}^{n}A_l(1+y) = \sum_{j=1}^{n}A_j(1+y)\sum_{k=1}^{n} C_k\sum_{l=1}^{n}A_l(1+y),$$

and

$$\begin{aligned}A_j(1+y)C_kA_l(1+y) &= \sum_{i=1}^{3}\delta_{i+3(j-1)}z^{j-1}(1+x^i)(1+y)z^{k-1}\left(\alpha_k(x+x^3)\right.\\ &\left.+\left[(\beta_k+\gamma_k)(1+x^3)+\gamma_k\hat{x}\right]y\right)\times\sum_{i=1}^{3}\delta_{i+3(l-1)}z^{l-1}(1+x^i)(1+y)\\ &= \sum_{i=1}^{3}\delta_{i+3(j-1)}z^{j-1}(1+x^i)(1+y)z^{k-1}\left((\beta_k+\gamma_k)(1+x^3)y\right)\\ &\times\sum_{i=1}^{3}\delta_{i+3(l-1)}z^{l-1}(1+x^i)(1+y)\end{aligned}$$

$$\begin{aligned}&= (\beta_k+\gamma_k)z^{j+k+l-3}\left(\delta_{3j-2}(1+x^2)(1+xy)+\delta_{3j-1}\hat{x}(1+y)\right.\\ &\left.+\delta_{3j}(1+x^2)(x+y)\right)\times\sum_{i=1}^{3}\delta_{i+3(l-1)}(1+x^i)(1+y)\\ &= (\beta_k+\gamma_k)z^{j+l-2}\left(\delta_{3j-2}(1+x^2)(1+xy)+\delta_{3j}(1+x^2)(x+y)\right)\\ &\times\sum_{i=1}^{3}\delta_{i+3(l-1)}(1+x^i)(1+y)\\ &= 0.\end{aligned}$$

Since $(1+xy)(1+x) = (1+x)(1+y)$, $(1+x^2)$ commutes with $(1+xy)$, $(1+x^2)(1+xy)(1+x^3) = \hat{x}(1+y)$, $(x+y)(1+x) = (1+x^2)(1+y)$, $(1+x^2)$ commutes with$(x+y)$ and $(x+y)(1+x^3) = (1+x)(1+y)$, therefore,

$$\begin{aligned}n^s &= 1+\sum_{j=1}^{n}F_j\hat{x}(1+y)+\sum_{k=1}^{n}C_k+\sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{k=1}^{n}C_k\\ &+\sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{l=1}^{n}A_l(1+y)+\sum_{k=1}^{n}C_k\sum_{l=1}^{n}A_l(1+y)\\ &= 1+\sum_{j=1}^{n}F_j\hat{x}(1+y)+\sum_{k=1}^{n}C_k+\sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{k=1}^{n}C_k\\ &+\sum_{j=1}^{n}A_j(1+y)\sum_{l=1}^{n}A_l(1+y)+\sum_{k=1}^{n}C_k\sum_{l=1}^{n}A_l(1+y)\\ &= 1+\sum_{k=1}^{n}C_k+\sum_{j=1}^{n}(A_j+F_j\hat{x})(1+y)\sum_{k=1}^{n}C_k+\sum_{k=1}^{n}C_k\sum_{l=1}^{n}A_l(1+y)\\ &= 1+\sum_{k=1}^{n}C_k+\sum_{j=1}^{n}A_j(1+y)\sum_{k=1}^{n}C_k+\sum_{k=1}^{n}C_k\sum_{l=1}^{n}A_l(1+y).\end{aligned}$$

Now, $(x+x^3)$ commutes with $(1+y)$, $A_j(1+y)\hat{x}y = \hat{x}yA_j(1+y) = 0$. Thus,

$$\begin{aligned} A_j(1+y)C_k + C_kA_j(1+y) &= (\beta_k+\gamma_k)z^{j+l-2}\left(\sum_{i=1}^{3}\delta_{i+3(j-1)}(1+x^i)(1+y)(1+x^3)y\right.\\ &\left.+(1+x^3)y\sum_{i=1}^{3}\delta_{i+3(j-1)}z^{j-1}(1+x^i)(1+y)\right)\\ &= (\beta_k+\gamma_k)z^{j+l-2}(\delta_{3j-2}+\delta_{3j})\hat{x}y\\ &= z^{j+l-2}(\delta_{3j-2}+\delta_{3j})(\beta_k+\gamma_k)\hat{x}y. \end{aligned}$$

It follows that $n^s \in N$. Similarly, $n^s \in N$ when n is even. Therefore, N is a normal subgroup of H. □

Theorem 2.1.

$$\mathcal{U}(\mathbb{F}_{2^k}(C_n\times D_8)) \cong \begin{cases} [(C_2^{nk}\times C_4^{nk})\rtimes(C_2^{nk}\times C_4^{nk})]\rtimes\mathcal{U}(\mathbb{F}_{2^k}(C_2\times C_n)) & \textit{when } n \textit{ is odd},\\ [(C_2^{2nk}\times C_4^{\frac{nk}{2}})\rtimes(C_2^{2nk}\times C_4^{\frac{nk}{2}})]\rtimes\mathcal{U}(\mathbb{F}_{2^k}(C_2\times C_n)) & \textit{when } n \textit{ is even}. \end{cases}$$

Proof. Clearly, $|H| = 2^{6nk}$, $|S| = |N| = 2^{3nk}$ and $S\cap N = \{1\}$. By the second Isomorphism Theorem, $H = NS$, therefore, $H \cong N \rtimes S$. This completes the proof. □

3 Conclusions

In this paper, we establish the structure of the unit group of the group algebra $\mathbb{F}_{2^k}(C_n\times D_8)$ for $n \geq 1$. Going forward, one could consider establishing the structure of the unit group of the group algebra $\mathbb{F}_{2^k}D_{2^n}$.

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